arXiv:hep-ph/0401048v2 31 Mar 2004

Decay widths and energy shifts of $\pi\pi$ and πK atoms

J. Schweizer

Institute for Theoretical Physics, University of Bern,
Sidlerstrasse 5, CH-3012 Bern, Switzerland
E-mail: schweizer@itp.unibe.ch

Abstract

We calculate the S-wave decay widths and energy shifts for $\pi^+\pi^-$ and $\pi^\pm K^\mp$ atoms in the framework of QCD + QED. The evaluation - valid at next-to-leading order in isospin symmetry breaking - is performed within a non-relativistic effective field theory. The results are of interest for future hadronic atom experiments.

Key words: Hadronic atoms, Chiral perturbation theory, Non relativistic effective Lagrangians, Isospin symmetry breaking, Electromagnetic corrections
PACS: 03.65.Ge, 03.65.Nk, 11.10.St, 12.39.Fe, 13.40.Ks

1 Introduction

Nearly fifty years ago, Deser *et al.* [1] derived the formulae for the decay width and strong energy shift of pionic hydrogen at leading order in isospin symmetry breaking. Similar relations also hold for $\pi^+\pi^-$ [2] and π^-K^+ atoms, which decay predominantly into $2\pi^0$ and π^0K^0, respectively. These Deser-type relations allow to extract the scattering lengths from measurements of the decay width and the strong energy shift. The DIRAC collaboration [3] at CERN intends to measure the lifetime of pionium in its ground state at the 10% level, which will allow to extract the scattering length difference $|a_0^0 - a_0^2|$ at 5% accuracy. The experimental result can then be compared with theoretical predictions for the S-wave scattering lengths [4–6] and with the results from other experiments [7]. Particularly interesting is the fact that one may determine in this manner the nature of the SU(2)×SU(2) spontaneous chiral symmetry breaking experimentally [8]. New experiments are proposed for CERN PS and J-PARC in Japan [9]. In order to determine the scattering lengths from such experiments, the theoretical expressions for the decay width and the strong energy shift must be known to an accuracy that matches the experimental

precision. For this reason, the ground state decay width of pionium has been evaluated at next-to-leading order [10–15] in the isospin symmetry breaking parameter δ, where both the fine-structure constant α and $(m_u - m_d)^2$ count as $\mathcal{O}(\delta)$. The aim of the present article is to provide the corresponding formulae for the S-wave decay widths and strong energy shifts of pionium and the $\pi^\pm K^\mp$ atom at next-to-leading order in isospin symmetry breaking. A detailed derivation of the results will be provided elsewhere [16]. The strong energy shift of the $\pi^\pm K^\mp$ atom is proportional to the sum of the isospin even and odd S-wave πK scattering lengths $a_0^+ + a_0^-$. This sum [18–22] is sensitive to the combination of low-energy constants $2L_6^r + L_8^r$ [23]. The consequences of this observation for the SU(3)×SU(3) quark condensate [24] remain to be worked out.

2 Non-relativistic framework

The non-relativistic effective Lagrangian framework has proven to be a very efficient method to investigate bound state characteristics [12,15,25,26]. The non-relativistic Lagrangian is exclusively determined by symmetries, which are rotational invariance, parity and time reversal. It provides a systematic expansion in powers of the isospin breaking parameter δ. What concerns the $\pi^- K^+$ atom, we count both α and $m_u - m_d$ as order δ. The different power counting for the $\pi^+\pi^-$ and $\pi^- K^+$ atoms are due to the fact that in QCD, the chiral expansion of the pion mass difference $\Delta_\pi = M_{\pi^+}^2 - M_{\pi^0}^2$ is of second order in $m_u - m_d$, while the kaon mass difference $\Delta_K = M_{K^+}^2 - M_{K^0}^2$ starts at first order in $m_u - m_d$. In the sector with one or two mesons, the non-relativistic πK Lagrangian is $\mathcal{L}_{\rm NR} = \mathcal{L}_1 + \mathcal{L}_2$. The first term contains the one-pion and one-kaon sectors,

$$\begin{aligned}\mathcal{L}_1 = {} & \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) + h_0^\dagger \Big(i\partial_t - M_{h^0} + \frac{\Delta}{2M_{h^0}} + \frac{\Delta^2}{8M_{h^0}^3} + \cdots \Big) h_0 \\ & + \sum_\pm h_\pm^\dagger \Big(iD_t - M_{h^+} + \frac{\mathbf{D}^2}{2M_{h^+}} + \frac{\mathbf{D}^4}{8M_{h^+}^3} + \cdots \Big) h_\pm, \end{aligned} \tag{1}$$

where $\mathbf{E} = -\nabla A_0 - \dot{\mathbf{A}}$, $\mathbf{B} = \nabla \times \mathbf{A}$ and the quantity $h = \pi, K$ stands for the non-relativistic pion and kaon fields. We work in the Coulomb gauge and eliminate the A^0 component of the photon field by the use of the equations of motion. The covariant derivatives are given by $D_t h_\pm = \partial_t h_\pm \mp ieA_0 h_\pm$ and $\mathbf{D} h_\pm = \nabla h_\pm \pm ie\mathbf{A} h_\pm$, where e denotes the electromagnetic coupling. What concerns the one-pion-one-kaon sector, we only list the terms needed to evaluate the decay width and the energy shift of the $\pi^- K^+$ atom at order $\delta^{9/2}$

and δ^4, respectively,

$$\mathcal{L}_2 = C_1' \pi_-^\dagger K_+^\dagger \pi_- K_+ + C_2 \left(\pi_-^\dagger K_+^\dagger \pi_0 K_0 + \text{h.c} \right) + C_3 \pi_0^\dagger K_0^\dagger \pi_0 K_0 + \cdots \quad (2)$$

The ellipsis stands for higher order terms [1]. We work in the center of mass system and thus omit terms proportional to the total 3-momentum. The total and reduced masses read

$$\Sigma_i = M_{\pi^i} + M_{K^i}, \quad \mu_i = \frac{M_{\pi^i} M_{K^i}}{M_{\pi^i} + M_{K^i}}, \quad i = +, 0. \quad (3)$$

The coupling constant C_1' contains contributions coming from the electromagnetic form factors of the pion and kaon,

$$C_1' = C_1 - e^2 \lambda, \quad \lambda = \frac{1}{6} \left(\langle r_{\pi^+}^2 \rangle + \langle r_{K^+}^2 \rangle \right), \quad (4)$$

where $\langle r_{\pi^+}^2 \rangle$ and $\langle r_{K^+}^2 \rangle$ denote the charge radii of the charged pion and kaon, respectively. The low energy constants $C_1, \ldots, C_3$ may be determined through matching the πK amplitude at threshold for various channels, see section 3.

To evaluate the energy shift and decay width of the $\pi^- K^+$ atom at next-to-leading order in isospin symmetry breaking, we make use of resolvents. For a detailed discussion of the technique, we refer to Ref. [15]. Here, we simply list the results. We use dimensional regularization, to treat both ultraviolet and infrared singularities. Up to and including order $\delta^{9/2}$, the decay into $\pi^0 K^0$ is the only decay channel contributing, and we get for the total S-wave decay width

$$\begin{aligned} \Gamma_n = & \frac{\alpha^3 \mu_+^3}{n^3 \pi^2} \mu_0 k_0 C_2^2 \left[1 - \frac{\mu_0^2 k_0^2 C_3^2}{4\pi^2} - \frac{\alpha \mu_+^2 C_1 \xi_n}{\pi} \right. \\ & \left. + \frac{5 \mu_0 k_0^2}{8} \frac{M_{\pi^0}^3 + M_{K^0}^3}{M_{\pi^0}^3 M_{K^0}^3} \right] + \mathcal{O}(\delta^5), \end{aligned} \quad (5)$$

where $k_0 = [2\mu_0(\Sigma_+ - \Sigma_0 - \alpha^2 \mu_+/(2n^2))]^{1/2}$ is of order $\delta^{1/2}$. The function ξ_n develops an ultraviolet singularity as $d \to 3$,

$$\begin{aligned} \xi_n = & \Lambda(\mu) - 1 + 2 \left[\ln \frac{\alpha}{n} + \ln \frac{2\mu_+}{\mu} + \psi(n) - \psi(1) - \frac{1}{n} \right], \\ \Lambda(\mu) = & \mu^{2(d-3)} \left[\frac{1}{d-3} - \ln 4\pi - \Gamma'(1) \right], \end{aligned} \quad (6)$$

[1] The basis of operators containing two space derivatives can be chosen such that none of them contributes to the energy shift and decay width at next-to-leading order in isospin symmetry breaking [16].

with $\psi(n) = \Gamma'(n)/\Gamma(n)$ and the running scale μ. At order δ^4, the total energy shift may be split into a strong part and an electromagnetic part, according to

$$\Delta E_n = \Delta E_n^{\rm h} + \Delta E_n^{\rm em}. \tag{7}$$

For the discussion of the electromagnetic energy shift, we refer to section 4. The strong S-wave energy shift reads

$$\Delta E_n^{\rm h} = -\frac{\alpha^3 \mu_+^3}{\pi n^3}\left[C_1 - \frac{\alpha\mu_+^2}{2\pi}C_1^2\xi_n - \frac{\mu_0^2 k_0^2}{4\pi^2}C_2^2 C_3\right] + \mathcal{O}(\delta^5). \tag{8}$$

The results for the decay width (5) and energy shift (8) are valid at next-to-leading order in isospin symmetry breaking.

3 Matching the low-energy constants

The coupling constants C_i can be determined through matching the non-relativistic and the relativistic amplitudes at threshold. The coupling C_3 is needed at order δ^0 only. However, we have to determine both C_1 and C_2 at next-to-leading order in isospin symmetry breaking. The relativistic amplitudes are related to the non-relativistic ones through

$$T_{\rm R}^{lm;ik}(\mathbf{q};\mathbf{p}) = 4\left[\omega_i(\mathbf{p})\omega_k(\mathbf{p})\omega_l(\mathbf{q})\omega_m(\mathbf{q})\right]^{\frac{1}{2}} T_{\rm NR}^{lm;ik}(\mathbf{q};\mathbf{p}), \tag{9}$$

with $\omega_i(\mathbf{p}) = (M_i^2 + \mathbf{p}^2)^{1/2}$. The 3-momentum $\mathbf{p}$ denotes the center of mass momentum of the incoming particles, $\mathbf{q}$ the one of the outgoing particles. The effective Lagrangian in Eqs. (1) and (2), allows us to evaluate the non-relativistic $\pi^- K^+ \to \pi^0 K^0$ and $\pi^- K^+ \to \pi^- K^+$ scattering amplitudes at threshold at order δ. In the isospin symmetry limit, the effective couplings C_1, C_2 and C_3 are

$$C_1 = \frac{2\pi}{\mu_+}\left(a_0^+ + a_0^-\right), \quad C_2 = -\frac{2\sqrt{2}\pi}{\mu_+}a_0^-, \quad C_3 = \frac{2\pi}{\mu_+}a_0^+, \tag{10}$$

where the S-wave scattering lengths[2] $a_0^+ = 1/3(a_0^{1/2} + 2a_0^{3/2})$ and $a_0^- = 1/3(a_0^{1/2} - a_0^{3/2})$ are defined in QCD, at $m_u = m_d$ and $M_\pi \doteq M_{\pi^+}$, $M_K \doteq M_{K^+}$. By substituting these relations into the expression for the decay width (5) and the strong energy shift (8), one obtains the Deser-type formulae [1,2]. We demonstrate the matching at next-to-leading order in δ by means of the $\pi^- K^+ \to \pi^- K^+$ amplitude. In the presence of virtual photons, we first have to subtract the one-photon exchange diagram from the full amplitude, as displayed in Fig. 1. The coupling constant C_1 is determined by the truncated part $\bar{T}_{\rm NR}^{\pm;\pm}$, which contains an infrared singular Coulomb phase θ_c as $d \to 3$,

[2] a_0^+ and a_0^- are normalized as in Ref. [18].

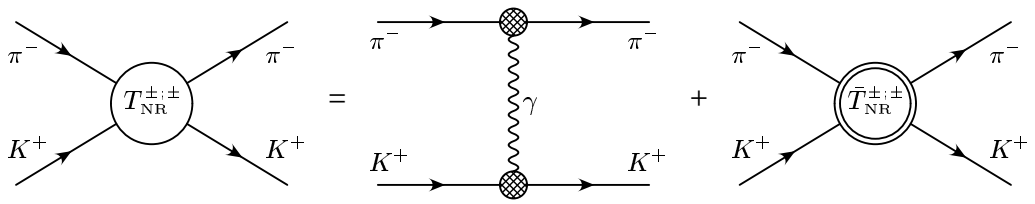

Fig. 1. Non-relativistic $\pi^- K^+ \to \pi^- K^+$ scattering amplitude. The blob describes the vector form factor of the pion and kaon. $\bar{T}^{\pm;\pm}_{\rm NR}$ denotes the truncated amplitude.

$$\bar{T}^{\pm;\pm}_{\rm NR}(\mathbf{p};\mathbf{p}) = e^{2i\alpha\theta_c}\hat{T}^{\pm;\pm}_{\rm NR}(\mathbf{p};\mathbf{p}),$$
$$\theta_c = \frac{\mu_+}{|\mathbf{p}|}\mu^{d-3}\left\{\frac{1}{d-3} - \frac{1}{2}\left[\ln 4\pi + \Gamma'(1)\right] + \ln\frac{2|\mathbf{p}|}{\mu}\right\}. \tag{11}$$

At order δ, the remainder $\hat{T}^{\pm;\pm}_{\rm NR}$ is free of infrared singularities at threshold. The real part of $\hat{T}^{\pm;\pm}_{\rm NR}$ is given by

$$\mathrm{Re}\,\hat{T}^{\pm;\pm}_{\rm NR}(\mathbf{p};\mathbf{p}) = \frac{B_1'}{|\mathbf{p}|} + B_2'\ln\frac{|\mathbf{p}|}{\mu_+} + \frac{1}{4M_{\pi^+}M_{K^+}}\mathrm{Re}\,A^{\pm;\pm}_{\rm thr} + \mathcal{O}(\mathbf{p}), \tag{12}$$

with $B_1' = C_1\alpha\pi\mu_+ + o(\delta)$, $B_2' = -C_1^2\alpha\mu_+^2/\pi + o(\delta)$ and

$$\frac{1}{4M_{\pi^+}M_{K^+}}\mathrm{Re}\,A^{\pm;\pm}_{\rm thr} = C_1\left\{1 + \frac{C_1\alpha\mu_+^2}{2\pi}\left[1 - \Lambda(\mu) - \ln\frac{4\mu_+^2}{\mu^2}\right]\right\}$$
$$-\frac{C_2^2 C_3\mu_0^3}{2\pi^2}(\Sigma_+ - \Sigma_0) + o(\delta). \tag{13}$$

Here, the ultraviolet pole term $\Lambda(\mu)$ is removed by renormalizing the coupling C_1. The renormalization of C_1 eliminates at the same time the ultraviolet divergence contained in the expression for the energy shift (8). The calculation of the relativistic $\pi^- K^+ \to \pi^- K^+$ scattering amplitude was performed at $\mathcal{O}(p^4, e^2p^2)$ in Refs. [20,21]. Both the Coulomb phase and the logarithmic singularity in Eq. (12) are absent in the real part of the relativistic amplitude at this order of accuracy, they first occur at order e^2p^4. The quantity $\mathrm{Re}\,A^{\pm;\pm}_{\rm thr}$ denotes the constant term occurring in the real part of the truncated relativistic threshold amplitude. The coupling constant C_2 may be determined analogously by matching the non-relativistic $\pi^- K^+ \to \pi^0 K^0$ amplitude to the relativistic one at order δ.

4 Results for the $\pi^- K^+$ atom

The result for the decay width and strong energy shift are valid at next-to-leading order in isospin symmetry breaking, and to all orders in the chiral

expansion. We get for the decay width at order $\delta^{9/2}$, in terms of the relativistic $\pi^- K^+ \to \pi^0 K^0$ threshold amplitude,

$$\Gamma_n = \frac{8}{n^3}\alpha^3 \mu_+^2 p_n^* \mathcal{A}^2 \left(1 + K_n\right), \quad \mathcal{A} = -\frac{1}{8\sqrt{2}\pi}\frac{1}{\Sigma_+}\mathrm{Re}\, A_{\rm thr}^{00;\pm} + o(\delta), \tag{14}$$

where

$$\begin{aligned} K_n = & \frac{M_{\pi^+}\Delta_K + M_{K^+}\Delta_\pi}{M_{\pi^+} + M_{K^+}}(a_0^+)^2 \\ & -4\alpha\mu_+(a_0^+ + a_0^-)\left[\psi(n) - \psi(1) - \frac{1}{n} + \ln\frac{\alpha}{n}\right] + o(\delta). \end{aligned} \tag{15}$$

The outgoing relative 3-momentum

$$p_n^* = \frac{1}{2E_n}\lambda\left(E_n^2, M_{\pi^0}^2, M_{K^0}^2\right)^{1/2}, \tag{16}$$

with $\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$, is chosen such that the total final state energy corresponds to $E_n = \Sigma_+ - \alpha^2\mu_+/(2n^2)$. The quantity $\mathrm{Re}\, A_{\rm thr}^{00;\pm}$ is calculated as follows. One evaluates the relativistic $\pi^- K^+ \to \pi^0 K^0$ amplitude near threshold and removes the divergent Coulomb phase. The real part contains singularities $\sim 1/|\mathbf{p}|$ and $\sim \ln|\mathbf{p}|/\mu_+$. The constant term in this expansion corresponds to $\mathrm{Re}\, A_{\rm thr}^{00;\pm}$. The normalization is chosen such that

$$\mathcal{A} = a_0^- + \epsilon. \tag{17}$$

The isospin breaking corrections ϵ have been evaluated at $\mathcal{O}(p^4, e^2p^2)$ in Refs. [21,27]. See also the comments in section 6.

We now discuss the various energy shift contributions. According to Eq. (7), the energy shift at order δ^4 is split into an electromagnetic part $\Delta E_n^{\rm em}$ and the strong part $\Delta E_n^{\rm h}$ in Eq. (8). The electromagnetic energy shift contains both pure QED corrections as well as finite size effects due to the charge radii of the pion and kaon, contained in λ. The pure electromagnetic corrections have been evaluated in Ref. [28] for arbitrary angular momentum l. We checked [3] that the electromagnetic energy shift at order α^4 indeed amounts to

$$\begin{aligned} \Delta E_{nl}^{\rm em} = & \frac{\alpha^4\mu_+}{n^3}\left(1 - \frac{3\mu_+}{\Sigma_+}\right)\left[\frac{3}{8n} - \frac{1}{2l+1}\right] + \frac{4\alpha^4\mu_+^3\lambda}{n^3}\delta_{l0} \\ & + \frac{\alpha^4\mu_+^2}{\Sigma_+}\left[\frac{1}{n^3}\delta_{l0} + \frac{1}{n^4} - \frac{3}{n^3(2l+1)}\right] + \mathcal{O}(\alpha^5\ln\alpha). \end{aligned} \tag{18}$$

[3] We thank A. Rusetsky for a very useful communication concerning technical aspects of the calculation.

Here, the first term is generated by the mass insertions, the second contains the finite size effects and the last stems from the one-photon exchange contribution. The strong S-wave energy shift reads at order δ^4,

$$\Delta E_n^{\rm h} = -\frac{2\alpha^3\mu_+^2}{n^3}\mathcal{A}'\,(1+K_n')\,, \quad \mathcal{A}' = \frac{1}{8\pi\Sigma_+}{\rm Re}\,A_{\rm thr}^{\pm;\pm} + o(\delta), \tag{19}$$

with

$$K_n' = -2\alpha\mu_+(a_0^+ + a_0^-)\left[\psi(n) - \psi(1) - \frac{1}{n} + \ln\frac{\alpha}{n}\right] + o(\delta). \tag{20}$$

In the isospin limit, the normalized relativistic amplitude

$$\mathcal{A}' = a_0^+ + a_0^- + \epsilon', \tag{21}$$

reduces to the sum of the isospin even and odd scattering lengths. The corrections ϵ' have been obtained at $\mathcal{O}(p^4, e^2p^2)$ in Refs. [20,21]. See also the comments in section 6. The result for $\Delta E_1^{\rm h}$ in Eq. (19) agrees with the one obtained for the strong energy shift of the ground state in pionic hydrogen [26], if we replace μ_+ with the reduced mass of the $\pi^- p$ atom and ${\rm Re}\,A_{\rm thr}^{\pm;\pm}$ with the constant term in the threshold expansion for the real part of the truncated $\pi^- p \to \pi^- p$ amplitude.

What remains to be added are the vacuum polarization contributions [14,29], which are formally of higher order in α, however numerically not negligible. The vacuum polarization leads to an energy level shift $\Delta E_{nl}^{\rm vac}$ as well as to a change in the Coulomb wave function of the $\pi^- K^+$ atom at the origin $\delta\psi_{K,n}(0)$. For the first two energy levels, $\Delta E_{nl}^{\rm vac}$ [14,29] is given numerically in table 2, section 6. Formally of order α^{2l+5}, this contribution is enhanced due to its large coefficient containing $(\mu_+/m_e)^{2l+2}$. The modified Coulomb wave function affects both, the decay width and the strong energy shift, see section 6.

As discussed in section 6, the electromagnetic contributions (18) are known to a high precision. Further, the strong shift in the nP state is very much suppressed (order α^5). A future measurement of the energy splitting between the nS and nP states will therefore allow to extract the strong S-wave energy shift in Eq. (19), and to determine the combination $a_0^+ + a_0^-$ of the πK scattering lengths. The energy splitting between the 2S and 2P states is given by

$$\begin{aligned}\Delta E_{2s-2p} &= \Delta E_2^{\rm h} + \Delta E_{20}^{\rm em} - \Delta E_{21}^{\rm em} + \Delta E_{20}^{\rm vac} - \Delta E_{21}^{\rm vac} \\ &= -1.4 \pm 0.1\,{\rm eV}.\end{aligned} \tag{22}$$

The uncertainty displayed is the one in $\Delta E_2^{\rm h}$ only. For the numerical values of the various energy shift contributions, see table 2 in section 6.

5 Results for pionium

The decay rate and strong energy shift of pionium can be obtained from the formulae in Eqs. (5) and (8) through the following substitutions of the masses $M_{K^+} \to M_{\pi^+}$, $M_{K^0} \to M_{\pi^0}$ and the coupling constants $C_1 \to c_1$, $C_2 \to \sqrt{2}(c_2 - 2c_4\Delta_\pi)$ and $C_3 \to 2c_3$ [16]. The c_i are the low-energy constants defined in Ref. [15]. The S-wave decay width of the $\pi^+\pi^-$ atom reads at order $\delta^{9/2}$, in terms of the relativistic $\pi^+\pi^- \to \pi^0\pi^0$ threshold amplitude,

$$\begin{aligned}
\Gamma_{\pi,n} &= \frac{2}{9n^3}\alpha^3 p^*_{\pi,n}\mathcal{A}_\pi^2\left(1 + K_{\pi,n}\right), \\
\mathcal{A}_\pi &= a_0^0 - a_0^2 + \epsilon_\pi, \\
K_{\pi,n} &= \frac{\kappa}{9}\left(a_0^0 + 2a_0^2\right)^2 - \frac{2\alpha}{3}\left(2a_0^0 + a_0^2\right)\left[\psi(n) - \psi(1) - \frac{1}{n} + \ln\frac{\alpha}{n}\right] + o(\delta), \\
p^*_{\pi,n} &= \left(\Delta_\pi - \frac{\alpha^2}{4n^2}M^2_{\pi^+}\right)^{1/2},
\end{aligned} \tag{23}$$

where $\kappa = M^2_{\pi^+}/M^2_{\pi^0} - 1$. The quantity $\mathcal{A}_\pi$ is defined as in Refs. [13,15]. The isospin symmetry breaking corrections ϵ_π have been evaluated at $\mathcal{O}(p^4, p^2e^2)$ in Refs. [13,15,30]. For the decay width of the ground state at order $\delta^{9/2}$, we reproduce the result obtained in Refs. [11,13,15]. The electromagnetic energy shift $\Delta E^{\rm em}_{\pi,nl}$ is obtained from Eq. (18) through the above mass substitutions and $\lambda \to 1/3\langle r^2_{\pi^+}\rangle$. Finally, the S-wave energy shift of the $\pi^+\pi^-$ atom reads at order δ^4, in terms of the relativistic one-particle irreducible $\pi^+\pi^- \to \pi^+\pi^-$ amplitude at threshold,

$$\begin{aligned}
\Delta E^{\rm h}_{\pi,n} &= -\frac{\alpha^3 M_{\pi^+}}{n^3}\mathcal{A}'_\pi\left(1 + K'_{\pi,n}\right), \\
\mathcal{A}'_\pi &= \frac{1}{6}\left(2a_0^0 + a_0^2\right) + \epsilon'_\pi, \\
K'_{\pi,n} &= -\frac{\alpha}{3}\left(2a_0^0 + a_0^2\right)\left[\psi(n) - \psi(1) - \frac{1}{n} + \ln\frac{\alpha}{n}\right] + o(\delta),
\end{aligned} \tag{24}$$

where $\mathcal{A}'_\pi$ is defined analogously to the quantity $\mathcal{A}'$ discussed in section 4. The isospin symmetry breaking contributions ϵ'_π have been calculated at $\mathcal{O}(e^2p^2)$ in Refs. [31,32]. For pionium the energy splitting between the 2S and 2P states reads

$$\begin{aligned}
\Delta E_{\pi,2s-2p} &= \Delta E^{\rm h}_{\pi,2} + \Delta E^{\rm em}_{\pi,20} - \Delta E^{\rm em}_{\pi,21} + \Delta E^{\rm vac}_{\pi,20} - \Delta E^{\rm vac}_{\pi,21} \\
&= -0.59 \pm 0.01\,{\rm eV}.
\end{aligned} \tag{25}$$

Again the uncertainty displayed is the one in $\Delta E^{\rm h}_{\pi,2}$ only. The numerical values for the various energy shifts are listed in table 3, section 6.

	$\delta_{h,1}$	$\delta'_{h,1}$	$\delta'_{h,2}$
$\pi^+\pi^-$ atom	$(5.8 \pm 1.2) \cdot 10^{-2}$	$(6.2 \pm 1.2) \cdot 10^{-2}$	$(6.1 \pm 1.2) \cdot 10^{-2}$
$\pi^\pm K^\mp$ atom	$(4.0 \pm 2.2) \cdot 10^{-2}$	$(1.7 \pm 2.2) \cdot 10^{-2}$	$(1.5 \pm 2.2) \cdot 10^{-2}$

Table 1
Next-to-leading order corrections to the Deser-type formulae.

6 Numerical analysis

For the S-wave $\pi\pi$ scattering lengths, we use the chiral predictions $a_0^0 = 0.220 \pm 0.005$ and $a_0^2 = -0.0444 \pm 0.0010$ [5,6]. The correlation matrix for a_0^0 and a_0^2 is given in Ref. [6]. For the isospin symmetry breaking corrections to the $\pi\pi$ threshold amplitudes (23) and (24), we use $\epsilon_\pi = (0.61 \pm 0.16) \cdot 10^{-2}$ and $\epsilon'_\pi = (0.37 \pm 0.08) \cdot 10^{-2}$ as given in Ref. [15] and [32], respectively. For the πK scattering lengths, we use the values from the recent analysis of data and Roy-Steiner equations [22], $a_0^+ = (0.045 \pm 0.012) M_{\pi^+}^{-1}$ and $a_0^- = (0.090 \pm 0.005) M_{\pi^+}^{-1}$. The correlation parameter for a_0^+ and a_0^- is given in Ref. [22]. The isospin breaking corrections to the πK threshold amplitudes (17) and (21) have been worked out in [20,21,27]. Whereas the analytic expressions for ϵ and ϵ' obtained in [20,21,27] are not identical, the numerical values agree within the uncertainties quoted in [21]. In the following, we use [21] $\epsilon = (0.1 \pm 0.1) \cdot 10^{-2} M_{\pi^+}^{-1}$ and $\epsilon' = (0.1 \pm 0.3) \cdot 10^{-2} M_{\pi^+}^{-1}$. For the charge radii of the pion and kaon, we take $\langle r_{\pi^+}^2 \rangle = (0.452 \pm 0.013)\,\mathrm{fm}^2$ and $\langle r_{K^+}^2 \rangle = (0.363 \pm 0.072)\,\mathrm{fm}^2$ [33].

We obtain for the decay width of the ground state,

$$\Gamma_1 = 8\alpha^3 \mu_+^2 p_1^*(a_0^-)^2 \,(1 + \delta_{K,1})\,, \quad \Gamma_{\pi,1} = \frac{2\alpha^3}{9} p_{\pi,1}^*(a_0^0 - a_0^2)^2 \,(1 + \delta_{\pi,1})\,, \tag{26}$$

where the corrections $\delta_{h,1}$, $h = \pi, K$ are given in table 1. The strong energy shift reads

$$\begin{aligned} \Delta E_n^{\mathrm{h}} &= -\frac{2\alpha^3 \mu_+^2}{n^3}(a_0^+ + a_0^-)\left(1 + \delta'_{K,n}\right), \\ \Delta E_{\pi,n}^{\mathrm{h}} &= -\frac{\alpha^3 M_{\pi^+}}{6n^3}(2a_0^0 + a_0^2)\left(1 + \delta'_{\pi,n}\right). \end{aligned} \tag{27}$$

For the first two energy levels, the corrections $\delta'_{h,n}$ are specified in table 1. As mentioned in section 4, these corrections to the Deser-type formulae are modified by vacuum polarization,

$$\delta_{h,n} \to \delta_{h,n} + \delta_{h,n}^{\mathrm{vac}}, \quad \delta'_{h,n} \to \delta'_{h,n} + \delta_{h,n}^{\mathrm{vac}}, \tag{28}$$

$\pi^{\pm}K^{\mp}$ atom	$\Delta E_{nl}^{\rm em}$[eV]	$\Delta E_{nl}^{\rm vac}$[eV]	$\Delta E_{n}^{\rm h}$[eV]	τ_n[s]
n=1, l=0	-0.095	-2.56	-9.0 ± 1.1	$(3.7 \pm 0.4) \cdot 10^{-15}$
n=2, l=0	-0.019	-0.29	-1.1 ± 0.1	
n=2, l=1	-0.006	-0.02		

Table 2
Numerical values for the energy shift and the lifetime of the $\pi^{\pm}K^{\mp}$ atom.

$\pi^{+}\pi^{-}$ atom	$\Delta E_{\pi,nl}^{\rm em}$[eV]	$\Delta E_{\pi,nl}^{\rm vac}$[eV]	$\Delta E_{\pi,n}^{\rm h}$[eV]	$\tau_{\pi,n}$[s]
n=1, l=0	-0.065	-0.942	-3.8 ± 0.1	$(2.9 \pm 0.1) \cdot 10^{-15}$
n=2, l=0	-0.012	-0.111	-0.47 ± 0.01	
n=2, l=1	-0.004	-0.004		

Table 3
Numerical values for the energy shift and the lifetime of the $\pi^{+}\pi^{-}$ atom.

where

$$\delta_{h,n}^{\rm vac} = \frac{2\delta\psi_{h,n}(0)}{\psi_{h,n}(0)}. \tag{29}$$

Formally, the contribution $\delta_{h,n}^{\rm vac}$ is of order α^2, but enhanced because of the large coefficient containing μ_+/m_e. For the ground state, the corrections [14] yield $\delta_{K,1}^{\rm vac} = 0.45 \cdot 10^{-2}$ and $\delta_{\pi,1}^{\rm vac} = 0.31 \cdot 10^{-2}$. The changes in $\delta_{\pi,1}$ and $\delta'_{\pi,1}$ due to $\delta_{\pi,1}^{\rm vac}$ are about 5%. For $\delta'_{K,1}$ however, the correction amounts to 27%. Here, we omit the contributions from $\delta_{h,n}^{\rm vac}$, because the uncertainties in $\delta_{h,n}$ and $\delta'_{h,n}$ are much larger than $\delta_{h,n}^{\rm vac}$.

The numerical values for the lifetime $\tau_1 \doteq \Gamma_1^{-1}$, ($\tau_{\pi,1} \doteq \Gamma_{\pi,1}^{-1}$) and the energy shifts at next-to-leading order in isospin symmetry breaking are given in table 2 and 3. The energy shifts due to vacuum polarization $\Delta E_{nl}^{\rm vac}$ are taken from Ref. [14,29]. In the evaluation of the uncertainties, the correlations between the S-wave scattering lengths have been taken into account. For the decay width and the strong energy shift of the $\pi^{\pm}K^{\mp}$ atom, the dominant source of uncertainty is due to the uncertainties in the scattering lengths a_0^+ and a_0^-. We do not display the error bars for the electromagnetic energy shifts, which stem at order α^4 from the uncertainties in $\langle r_{\pi^+}^2 \rangle$ and $\langle r_{K^+}^2 \rangle$ only. For pionium, the uncertainties of $\Delta E_{\pi,10}^{\rm em}$ at order α^4 amount to about 0.7%, while for the $\pi^{\pm}K^{\mp}$ atom $\Delta E_{10}^{\rm em}$ is known at the 5% level. To estimate the order of magnitude of the electromagnetic corrections at higher order, we may compare with positronium. Here, the α^5 and $\alpha^5 \ln\alpha$ corrections [34] amount to about 2% with respect to the α^4 contributions.

7 Summary and Conclusions

We provided the formulae for the energy shifts and decay widths of the $\pi^+\pi^-$ and $\pi^\pm K^\mp$ atoms at next-to-leading order in isospin symmetry breaking. To confront these predictions with data presents a challenge for future hadronic atom experiments. Should it turn out that these predictions are in conflict with experiment, one would have to revise our present understanding of the low-energy structure of QCD.

Acknowledgments

It is a great pleasure to thank J. Gasser for many interesting discussions and suggestions. Further, I thank R. Kaiser, A. Rusetsky, H. Sazdjian and J. Schacher for useful discussions as well as for helpful comments on the manuscript. This work was supported in part by the Swiss National Science Foundation and by RTN, BBW-Contract N0. 01.0357 and EC-Contract HPRN–CT2002–00311 (EURIDICE).

References

[1] S. Deser, M. L. Goldberger, K. Baumann and W. Thirring, Phys. Rev. **96** (1954) 774; T.L. Trueman, Nucl. Phys. **26** (1961) 57.

[2] T. R. Palfrey and J. L. Uretsky, Phys. Rev. **121** (1961) 1798; S. M. Bilenky, V. H. Nguyen, L. L. Nemenov and F. G. Tkebuchava, Yad. Fiz. **10** (1969) 812.

[3] B. Adeva *et al.*, CERN-SPSLC-95-1.

[4] S. Weinberg, Phys. Rev. Lett. **17** (1966) 616; J. Gasser and H. Leutwyler, Phys. Lett. B **125** (1983) 321; J. Bijnens, G. Colangelo, G. Ecker, J. Gasser and M. E. Sainio, Phys. Lett. B **374** (1996) 210 [arXiv:hep-ph/9511397]; J. Bijnens, G. Colangelo, G. Ecker, J. Gasser and M. E. Sainio, Nucl. Phys. B **508** (1997) 263 [Erratum-ibid. B **517** (1998) 639] [arXiv:hep-ph/9707291]; J. Nieves and E. Ruiz Arriola, Eur. Phys. J. A **8** (2000) 377 [arXiv:hep-ph/9906437]; G. Amoros, J. Bijnens and P. Talavera, Nucl. Phys. B **585** (2000) 293 [Erratum-ibid. B **598** (2001) 665] [arXiv:hep-ph/0003258].

[5] G. Colangelo, J. Gasser and H. Leutwyler, Phys. Lett. B **488** (2000) 261 [arXiv:hep-ph/0007112].

[6] G. Colangelo, J. Gasser and H. Leutwyler, Nucl. Phys. B **603** (2001) 125 [arXiv:hep-ph/0103088].

[7] L. Rosselet *et al.*, Phys. Rev. D **15** (1977) 574; C. D. Froggatt and J. L. Petersen, Nucl. Phys. B **129**, 89 (1977); M. M. Nagels *et al.*, Nucl. Phys. B **147** (1979) 189; A. A. Bolokhov, M. V. Polyakov and S. G. Sherman, Eur. Phys. J. A **1** (1998) 317 [arXiv:hep-ph/9707406]; D. Pocanic, arXiv:hep-ph/9809455; CHAOS Collaboration, J. B. Lange *et al.*, Phys. Rev. Lett. **80**, 1597 (1998); J. B. Lange *et al.*, Phys. Rev. C **61** (2000) 025201; NA48 Collaboration, R. Batley *et al.*, CERN-SPSC-2000-003; S. Pislak *et al.* [BNL-E865 Collaboration], Phys. Rev. Lett. **87** (2001) 221801 [arXiv:hep-ex/0106071]; S. Descotes-Genon, N. H. Fuchs, L. Girlanda and J. Stern, Eur. Phys. J. C **24** (2002) 469 [arXiv:hep-ph/0112088]; S. Pislak *et al.*, Phys. Rev. D **67** (2003) 072004 [arXiv:hep-ex/0301040]; V. N. Maiorov and O. O. Patarakin, arXiv:hep-ph/0308162.

[8] M. Knecht, B. Moussallam, J. Stern and N. H. Fuchs, Nucl. Phys. B **457** (1995) 513 [arXiv:hep-ph/9507319]; M. Knecht, B. Moussallam, J. Stern and N. H. Fuchs, Nucl. Phys. B **471** (1996) 445 [arXiv:hep-ph/9512404]; L. Girlanda, M. Knecht, B. Moussallam and J. Stern, Phys. Lett. B **409** (1997) 461 [arXiv:hep-ph/9703448].

[9] B. Adeva *et al.*, CERN-SPSC-2000-032; L. Afanasyev *et al.*, Lifetime measurement of $\pi^+\pi^-$ and $\pi^\pm K^\mp$ atoms to test low energy QCD (2002), letter of intent of the experiment at Japan Proton Accelerator Research Complex; available at http://dirac.web.cern.ch/DIRAC/future.html.

[10] Z. K. Silagadze, JETP Lett. **60** (1994) 689 [arXiv:hep-ph/9411382]; A. Gashi, G. Rasche, G. C. Oades and W. S. Woolcock, Nucl. Phys. A **628** (1998) 101 [arXiv:nucl-th/9704017]; Z. K. Silagadze, arXiv:hep-ph/9803307; M. A. Ivanov, V. E. Lyubovitskij, E. Z. Lipartia and A. G. Rusetsky, Phys. Rev. D **58** (1998) 094024 [arXiv:hep-ph/9805356]; P. Minkowski, in: Proceedings of the International Workshop *Hadronic Atoms and Positronium in the Standard Model*, Dubna, 26-31 May 1998, M. A. Ivanov *at al.*, Eds., Dubna 1998, p. 74 [arXiv:hep-ph/9808387]; E. A. Kuraev, Phys. Atom. Nucl. **61** (1998) 239; U. Jentschura, G. Soff, V. Ivanov and S. G. Karshenboim, Phys. Lett. A **241** (1998) 351; H. W. Hammer and J. N. Ng, Eur. Phys. J. A **6** (1999) 115 [arXiv:hep-ph/9902284].

[11] H. Jallouli and H. Sazdjian, Phys. Rev. D **58** (1998) 014011 [Erratum-ibid. D **58** (1998) 099901] [arXiv:hep-ph/9706450]; H. Sazdjian, Phys. Lett. B **490** (2000) 203 [arXiv:hep-ph/0004226].

[12] P. Labelle and K. Buckley, arXiv:hep-ph/9804201; X. Kong and F. Ravndal, Phys. Rev. D **59** (1999) 014031; Phys. Rev. D **61** (2000) 077506 [arXiv:hep-ph/9905539]; B. R. Holstein, Phys. Rev. D **60** (1999) 114030 [arXiv:nucl-th/9901041]; A. Gall, J. Gasser, V. E. Lyubovitskij and A. Rusetsky, Phys. Lett. B **462** (1999) 335 [arXiv:hep-ph/9905309]; D. Eiras and J. Soto, Phys. Rev. D **61** (2000) 114027 [arXiv:hep-ph/9905543]; V. Antonelli, A. Gall, J. Gasser and A. Rusetsky, Annals Phys. **286** (2001) 108 [arXiv:hep-ph/0003118].

[13] J. Gasser, V. E. Lyubovitskij and A. Rusetsky, Phys. Lett. B **471** (1999) 244 [arXiv:hep-ph/9910438].

[14] D. Eiras and J. Soto, Phys. Lett. B **491** (2000) 101 [arXiv:hep-ph/0005066].

[15] J. Gasser, V. E. Lyubovitskij, A. Rusetsky and A. Gall, Phys. Rev. D **64** (2001) 016008 [arXiv:hep-ph/0103157].

[16] J. Schweizer, in preparation. The calculation of the same quantities in the framework of constrained Bethe-Salpeter equation is in progress [17].

[17] H. Sazdjian, private communication.

[18] V. Bernard, N. Kaiser and U. G. Meissner, Phys. Rev. D **43** (1991) 2757.

[19] A. Roessl, Nucl. Phys. B **555** (1999) 507 [arXiv:hep-ph/9904230].

[20] A. Nehme, Eur. Phys. J. C **23** (2002) 707 [arXiv:hep-ph/0111212]; B. Kubis and U. G. Meissner, Phys. Lett. B **529** (2002) 69 [arXiv:hep-ph/0112154].

[21] B. Kubis, Dissertation Bonn Univ. (2002). Berichte des Forschungszentrums Jülich, **4007** ISSN 0944-2952.

[22] P. Buettiker, S. Descotes-Genon and B. Moussallam, arXiv:hep-ph/0310283.

[23] J. Gasser and H. Leutwyler, Nucl. Phys. B **250** (1985) 465.

[24] S. Descotes-Genon, L. Girlanda and J. Stern, JHEP **0001** (2000) 041 [arXiv:hep-ph/9910537]; B. Moussallam, JHEP **0008** (2000) 005 [arXiv:hep-ph/0005245]; S. Descotes-Genon, L. Girlanda and J. Stern, Eur. Phys. J. C **27** (2003) 115 [arXiv:hep-ph/0207337]; S. Descotes-Genon, N. H. Fuchs, L. Girlanda and J. Stern, arXiv:hep-ph/0311120.

[25] W. E. Caswell and G. P. Lepage, Phys. Lett. B **167** (1986) 437.

[26] V. E. Lyubovitskij and A. Rusetsky, Phys. Lett. B **494** (2000) 9 [arXiv:hep-ph/0009206].

[27] B. Kubis and U. G. Meissner, Nucl. Phys. A **699** (2002) 709 [arXiv:hep-ph/0107199]; A. Nehme and P. Talavera, Phys. Rev. D **65** (2002) 054023 [arXiv:hep-ph/0107299].

[28] I. T. Todorov, Phys. Rev. D **3** (1971) 2351; A. Nandy, Phys. Rev. D **5** (1972) 1531; H. Jallouli and H. Sazdjian, Annals Phys. **253** (1997) 376 [arXiv:hep-ph/9602241].

[29] G. E. Pustovalov, JETP **5** (1957) 1234; S. G. Karshenboim, Can. J. Phys. **76** (1998) 169; S. G. Karshenboim, U. D. Jentschura, V. G. Ivanov and G. Soff, Eur. Phys. J. D **2** (1998) 209.

[30] M. Knecht and R. Urech, Nucl. Phys. B **519** (1998) 329 [arXiv:hep-ph/9709348].

[31] U. G. Meissner, G. Muller and S. Steininger, Phys. Lett. B **406** (1997) 154 [Erratum-ibid. B **407** (1997) 454] [arXiv:hep-ph/9704377].

[32] M. Knecht and A. Nehme, Phys. Lett. B **532** (2002) 55 [arXiv:hep-ph/0201033].

[33] J. Bijnens and P. Talavera, JHEP **0203** (2002) 046 [arXiv:hep-ph/0203049]; see also S. R. Amendolia *et al.* [NA7 Collaboration], Nucl. Phys. B **277** (1986) 168; G. Colangelo, arXiv:hep-ph/0312017.

[34] R. Karplus and A. Klein, Phys. Rev. **87** (1952) 848.